A Simple Statical Approach to the Measurement of the Elastic Constants in Anisotropic Media

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A method is given whereby the twenty-one independent elastic constants in the general linear homogeneous anisotropic elastic body may be measured in a simple systematic manner by using a static homogeneous deformation field called "pure extension". Two types of test are used throughout.

1. Introduction

Classical linear elasticity theory is assumed to describe adequately the deformations of anisotropic bodies which are subject to small strains and small rotations. In general the elastic properties of a homogeneous elastic body are described in terms of twenty-one elastic constants. Clearly, to obtain these constants, twenty-one measurements are needed in general. It is the purpose of this paper to discuss what constitute appropriate static tests and measurements, and to put forward what I think is a simple scheme for the determination of the elastic constants.

The experimentalist has at his disposal the surface tractions which he may vary at will on a specimen. However, in so varying them, he will not be able to produce a given arbitrary deformation in a specimen of arbitrary elastic symmetry. Of course, if the material has an appropriate symmetry, he may be able to produce a desired deformation in it, but this will not be true in general. This is clearly seen. If the deformation is inhomogeneous, then in general the stresses are functions of position, and the equilibrium equations cannot be satisfied unless suitable body forces are supplied. It is proven formally here that the only deformations which can be maintained in every homogeneous elastic body by the action of surface forces alone are necessarily homogeneous. However, particular inhomogeneous deformations can be maintained in some bodies. For example, simple torsion can be maintained in an isotropic body by surface forces alone. The experimentalist does not know *a priori* what elastic symmetries a specimen may have. Accordingly it is pointless for him to endeavour to maintain anything other than a homogeneous deformation in the specimen by the application of surface forces alone.

Thus, in the main, this paper is concerned with a class of homogeneous deformations suitable for the determination of the elastic constants by means of static tests. This class of homogeneous deformations is particularly useful in that only two types of specimens are used throughout. This means that essentially the same two pieces of apparatus can be used for all the tests.

We now proceed with the theory.

In the classical linear elasticity theory of homogeneous bodies the stress-deformation relation takes the form†

$$t_{ij} = C_{ijkl}\ \partial u_k/\partial X_l \qquad (1)$$

all quantities being referred to a fixed rectangular Cartesian coordinate system x whose origin is 0. In this system the components of the (symmetric) Cauchy stress tensor are denoted by t_{ij} and the components of the elasticity tensor by C_{ijkl}. The displacement $\mathbf{u}$ has components u_i given by

$$\mathbf{u} = \mathbf{x} - \mathbf{X}\ ;\ u_i(\mathbf{X}) = x_i - X_i\,, \qquad (2)$$

where $\mathbf{x}$ is the position vector of the particle initially at $\mathbf{X}$. The elastic constants satisfy the symmetry conditions

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}\,. \qquad (3)$$

†Latin suffixes range over 1, 2, 3. Repeated suffixes are summed.

As a result there are at most twenty-one independent elastic constants.

Using equation 3, it is seen that equation 1 may be written

$$t_{ij} = C_{ijkl}\, e_{kl}\,, \tag{4}$$

where the strains e_{ij} are given by

$$2e_{ij} = \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\,. \tag{5}$$

For a homogenous classical linear elastic body the stress-strain relation is given by equation 4 where C_{ijkl} are constants which satisfy equation 3. In the absence of body forces the equilibrium equations

$$\frac{\partial t_{ij}}{\partial X_j} = C_{ijkl}\, \frac{\partial^2 u_k}{\partial X_l \partial X_j} = 0 \tag{6}$$

must be satisfied. The traction across a surface whose unit normal is **n** has components $t(\mathbf{n})_i$ given by

$$t(\mathbf{n})_i = n_j\, t_{ij} = n_j\, C_{ijkl}\, e_{kl}\,. \tag{7}$$

The present work is concerned with the measurement of the twenty-one elastic constants. The purpose is to show the merits of a static plane homogeneous deformation field, which I called "pure extension" [1], in making a systematic approach to the measurement of the elastic constants by means of static tests. The mathematics involved in pure extension is very easy, yet it provides the key in a very simple way on what experiments need to be performed.

Of course homogeneous deformations are the simplest to treat from a mathematical point of view. However there is a more compelling reason for using homogeneous deformations in the general theory of linear anisotropic bodies, for they are the only deformations which can be maintained in *every* body by the action of surface forces alone. To maintain other than homogeneous deformations in an arbitrary body would in general require the application of body forces. In fact the following analogue of Ericksen's Theorem [2] can be proved.

Theorem The only deformations which can be maintained in every homogeneous compressible linear anisotropic elastic solid with stress-deformation relation (equation 1), under the action of surface forces alone, is necessarily homogeneous.

Proof Three particular choices of C_{ijkl} will indicate the method. The deformation field which is sought can be maintained in every body by the action of surface forces alone. In particular it must be possible to maintain it by surface forces alone in the body for which $C_{1212} \neq 0$, $C_{ijkl} = 0$ otherwise. Then $t_{12} = C_{1212}l_{12}$, $t_{ij} = 0$, $ij \neq 12$. Hence the deformation must satisfy the equilibrium equations (6) which now give†

$$\frac{\partial e_{12}}{\partial X_2} = \frac{\partial e_{12}}{\partial X_1} = 0\,. \tag{8}$$

Next let $C_{3312} \neq 0$, $C_{ijkl} = 0$ otherwise. Then from equation 4 the only non-zero stresses are $t_{12} = C_{3312}\, e_{33}$, $t_{33} = C_{3312}\, e_{12}$. Again, the deformation must satisfy the corresponding equilibrium equations which give in this case,

$$\frac{\partial e_{12}}{\partial X_3} = 0\,,\ \frac{\partial e_{33}}{\partial X_1} = \frac{\partial e_{33}}{\partial X_2} = 0\,. \tag{9}$$

Finally let C_{3333} be the only non-zero component of C_{ijkl}. Then the only non-zero stress component is $t_{33} = C_{3333}\, e_{33}$. The deformation must now satisfy

$$\frac{\partial e_{33}}{\partial X_3} = 0\,. \tag{10}$$

Hence from equations 8, 9, and 10, e_{12} and e_{33} are both constants. In a similar way it can be shown that the remaining strains e_{ij} must also be constant. It follows immediately that the deformation is homogeneous and the theorem is proved.

In the next section pure extension is described. Then its use in measuring the elastic constants is indicated.

2. Pure Extension

The deformation field called "pure extension" is given by

$$u_i = \gamma A_i B_j X_j\,,\ A_i A_i = B_i B_i = 1\,. \tag{11}$$

The unit vectors **A** and **B** emanate from 0, the origin of the co-ordinate system, and γ is a constant chosen so small that the displacement field (equation 11) remains within the domain of the linear theory. The stress field associated with equation 11 is given by

$$t_{ij} = \gamma\, C_{ijkl}\, A_k\, B_l\,, \tag{12}$$

†This result may also be obtained by observing that if the deformation is to be possible for all C_{ijkl}, then in particular it must be possible both for C_{ijkl} and C^*_{ijkl} where $C^*_{ijkl} = C_{ijkl}$, $ijkl \neq 1212$, $C^*_{1212} = \lambda C_{1212}$ where λ is any constant. One could then write down the equilibrium equations using C_{ijkl} and C^*_{ijkl}. By subtracting corresponding equations, equation 8 is obtained.

and since these stresses are uniform, the equilibrium equations 6 are satisfied.

The geometrical description of pure extension given in [1] is repeated here.

Let **A**, **B** be fixed and consider first the case when $\mathbf{A} . \mathbf{B} \neq 0$. Then let **D** and **E** be any pair of unit vectors such that **A**, **D** and **E** form a right-handed triad of mutually perpendicular unit vectors. From equation 11, since $\mathbf{x} = \mathbf{X} + \mathbf{u}$, it follows that

$$(\mathbf{x} . \mathbf{D})^2 + (\mathbf{x} . \mathbf{E})^2 = (\mathbf{X} . \mathbf{D})^2 + (\mathbf{X} . \mathbf{E})^2 ,$$
$$\mathbf{x} . \mathbf{B} = \mathbf{X} . \mathbf{B} (1 + \gamma \mathbf{A} . \mathbf{B}) .$$

Thus the body (hereafter referred to as "the specimen") cut off by the circular cylinder of radius a, $(\mathbf{X} . \mathbf{D})^2 + (\mathbf{X} . \mathbf{E})^2 = a^2$, and by the parallel planes $\mathbf{X} . \mathbf{B} = 0$, $\mathbf{X} . \mathbf{B} = p$, is deformed into the similar body cut off by the circular cylinder $(\mathbf{x} . \mathbf{D})^2 + (\mathbf{x} . \mathbf{E})^2 = a^2$, and by the parallel planes $\mathbf{x} . \mathbf{B} = 0, \mathbf{x} . \mathbf{B} = p(1 + \gamma \mathbf{A}.\mathbf{B})$. The plane ends of the specimen remain plane and have the same area throughout the deformation. Each plane section whose normal is **B** remains plane and is shifted by an amount $k\gamma \mathbf{A} . \mathbf{B}$ parallel to itself in the direction of **A**, where k is the perpendicular distance of the plane section from 0. Hence for $\gamma \mathbf{A} . \mathbf{B} > 0$ we have what might be called "pure extension" and for $\gamma \mathbf{A} . \mathbf{B} < 0$ what might be called "pure compression". When $\mathbf{A} . \mathbf{B} = 1$, $\gamma > 0$, we have simple extension, and when $\mathbf{A} . \mathbf{B} = 1$, $\gamma < 0$, simple compression, in the usual terminology.

When $\mathbf{A} . \mathbf{B} = 0$, it is easy to see that equation 11 describes the simple shear of a unit cube whose faces are

$$\mathbf{A} . \mathbf{X} = 0, 1 \; ; \; \mathbf{B} . \mathbf{X} = 0, 1 \; ; \; (\mathbf{A} \wedge \mathbf{B}) . \mathbf{X} = 0, 1 .$$

The cube is sheared by an amount γ.

Hence equation 11 may be interpreted as a description of

$$\left\{ \begin{array}{l} \text{simple shear if } \mathbf{A} . \mathbf{B} = 0 \; ; \\ \text{simple extension or compression if } \mathbf{A} . \mathbf{B} = \pm 1 \; ; \\ \text{pure extension or compression if } \mathbf{A} . \mathbf{B} \neq 0, \; \mathbf{A} . \mathbf{B} \neq \pm 1 . \end{array} \right. \qquad (13)$$

In the next section particular choices of **A** and **B** will be made, thus fixing the deformation field (equation 11) up to a choice of γ. The corresponding specimen is determined by referring to equation 13.

For example, suppose $\mathbf{A} = (1, 0, 0)$, $\mathbf{B} = (-1, 0, 0)$. From equation 11 the displacement field is

$$u_1 = -\gamma X_1 , \; u_2 = 0 , \; u_3 = 0 .$$

Note that $\mathbf{A} . \mathbf{B} = -1$, so it is seen by referring to the table, 13, that the displacement describes simple extension or compression along the x_1 axis.

As another example, suppose $\mathbf{A} = (0, 1, 0)$, $\sqrt{2}\mathbf{B} = (1, 0, -1)$. Then from equation 11

$$u_1 = 0 , \; \sqrt{2}\, u_2 = X_1 - X_3 , \; u_3 = 0 .$$

Since $\mathbf{A} . \mathbf{B} = 0$, this displacement describes the simple shear of a cube whose faces are $X_2 = 0$, $X_2 = 1$; $X_1 - X_3 = 0, \sqrt{2}$; $X_1 + X_3 = 0, -\sqrt{2}$.

Note from equation 12 that

$$t_{ij}\, A_i B_j = \gamma\, C_{ijkl}\, A_i\, A_k\, B_l\, B_j . \qquad (14)$$

In view of equation 7 this may be written

$$\mathbf{t}_{(\mathbf{B})} . \mathbf{A} = \gamma\, C_{ijkl}\, A_i\, A_k\, B_l\, B_j . \qquad (15)$$

Now $\mathbf{t}_{(\mathbf{B})}$ is the traction across the surface whose unit outward normal is **B**, and so $\mathbf{t}_{(\mathbf{B})} . \mathbf{A}$ is the component along the axis of the specimen of the traction over its plane end.

In the case of simple shear $\mathbf{A} . \mathbf{B} = 0$. Then $\mathbf{t}_{(\mathbf{B})} . \mathbf{A}$ is the component of stress in the direction of the shear acting along the face $\mathbf{B} . \mathbf{X} = 1$.

Thus if **A** and **B** are fixed, the displacement is given by equation 11 where the magnitude and sign of γ are still undetermined. The shape of the specimen is determined on using the table 13. When the specimen shape is known the deformation of the requisite type may be produced in it, and $\mathbf{t}_{(\mathbf{B})} . \mathbf{A}$ and γ measured. The expression (equation 15) then connects these measurements with a linear combination of the elastic constants.

3. Measuring the Elastic Constants

For convenience the twenty-one independent elastic constants C_{ijkl} may be put into six groups as follows. (i) C_{1111}, C_{2222}, C_{3333}; (ii) C_{1212}, C_{2323}, C_{3131}; (iii) C_{1122}, C_{2233}, C_{3311}; (iv) C_{1112}, C_{1113}, C_{2223}, C_{2221}, C_{3331}, C_{3332}; (v) C_{1213}, C_{2321}, C_{3132}; (vi) C_{1123}, C_{2231}, C_{3312}.

The first elastic constant in each of these groups will be taken in turn, and it will be shown how, by appropriate choice of **A** and **B**, an appropriate experiment may be found through which the elastic constant may be measured. A choice of **A** and **B** that may be made in order to determine the remaining elements in each of these groups will also be given. Throughout it is assumed that the results of prior experiments may be used in later ones.

The key to the whole process is equation 15 and the table 13. **A** and **B** are chosen in such a way that the expression $C_{ijkl} A_i A_k B_l B_j$ involves the desired elastic constant, but involves as few as possible of the other elastic constants. Then the displacement is determined from equation 11, the specimen from the table 13 and $\mathbf{t}_{(\mathbf{B})}$. **A** and γ are measured experimentally.

(i) Take $\mathbf{A} = \mathbf{B} = (1, 0, 0)$. From equation 11

$$u_1 = \gamma X_1 , \; u_2 = 0 , \; u_3 = 0 .$$

The specimen is a circular cylindrical rod with axis along the x_1 axis and bounded by the planes $X_1 = 0$ and $X_1 = l$ (say). Now $A_i = B_i = \delta_{i1}$ and so

$$t_{ij} A_i B_j = t_{ij} \delta_{i1} \delta_{j1} = t_{11} ,$$
$$C_{ijkl} A_i A_k B_l B_j = C_{ijkl} \delta_{i1} \delta_{k1} \delta_{l1} \delta_{j1} = C_{1111} .$$

Hence from equation 15

$$t_{11} = \gamma C_{1111} .$$

Accordingly, by measuring t_{11} and γ, the value of C_{1111} can be determined.

A set of values of **A** and **B** suitable for the determination of C_{2222} and C_{3333} is

$$C_{2222} : \mathbf{A} = \mathbf{B} = (0, 1, 0) ,$$
$$C_{3333} : \mathbf{A} = \mathbf{B} = (0, 0, 1) .$$

(ii) To determine C_{1212}, let $\mathbf{A} = (1, 0, 0)$, $\mathbf{B} = (0, 1, 0)$. Note $\mathbf{A} . \mathbf{B} = 0$. Then from equation 11

$$u_1 = \gamma X_2 , \; u_2 = u_3 = 0$$

which describes the simple shear of the cube whose faces are $X_1 = 0, 1$; $X_2 = 0, 1$; $X_3 = 0, 1$. Now $A_i = \delta_{i1}$, $B_i = \delta_{i2}$ so that

$$C_{ijkl} A_i A_k B_l B_j = C_{ijkl} \delta_{i1} \delta_{k1} \delta_{l2} \delta_{j2} = C_{1212} ,$$
$$t_{ij} A_i B_j = t_{12} .$$

From equation 15

$$t_{12} = \gamma C_{1212} ,$$

and hence, by measuring the shear stress t_{12} and the amount of shear γ, C_{1212} is calculated.

A set of values of **A** and **B** suitable for the determination of C_{2323} and C_{3131} is

$$C_{2323} : \mathbf{A} = (0, 1, 0) , \; \mathbf{B} = (0, 0, 1) ,$$
$$C_{3131} : \mathbf{A} = (0, 0, 1) , \; \mathbf{B} = (1, 0, 0) .$$

(iii) To obtain C_{1122} let $\sqrt{2}\, \mathbf{A} = (1, 1, 0)$, $\sqrt{2}\, \mathbf{B} = (1, -1, 0)$, $\mathbf{A} . \mathbf{B} = 0$. From equation 11 the corresponding simple shear is given by

$$\sqrt{2}\, u_1 = \gamma (X_1 - X_2) , \; \sqrt{2}\, u_2 = \gamma (X_1 - X_2) , \; u_3 = 0 .$$

The specimen is the cube whose faces are

$X_1 + X_2 = 0, \sqrt{2}$; $X_1 - X_2 = 0 , \sqrt{2}$; $X_3 = 0, 1$.

Now

$$4 C_{ijkl} A_i A_k B_l B_j = C_{ijkl} (\delta_{i1} + \delta_{i2}) (\delta_{k1} + \delta_{k2}) (\delta_{l1} - \delta_{l2}) (\delta_{j1} - \delta_{j2})$$
$$= (C_{1j1l} + C_{1j2l} + C_{2j1l} + C_{2j2l}) (\delta_{l1} - \delta_{l2}) (\delta_{j1} - \delta_{j2})$$
$$= C_{1111} - 2C_{1122} + C_{2222} .$$

Thus from equation 15

$$4\mathbf{t}_{(\mathbf{B})} . \mathbf{A} = \gamma (C_{1111} + C_{2222}) - 2\gamma C_{1122} .$$

On measuring γ and $t_{(\mathbf{B})} . \mathbf{A}$, and using the results of (i), the elastic constant C_{1122} is determined.

A set of values of **A** and **B** suitable for the determination of C_{2233} and C_{3311} is

$$C_{2233} : \sqrt{2}\, \mathbf{A} = (0, 1, 1) , \; \sqrt{2}\, \mathbf{B} = (0, 1, -1)$$
$$C_{3311} : \sqrt{2}\, \mathbf{A} = (1, 0, 1) , \; \sqrt{2}\, \mathbf{B} = (1, 0, -1) .$$

(iv) To obtain C_{1112} let $\mathbf{A} = (1, 0, 0)$, $\sqrt{2}\, \mathbf{B} = (1, 1, 0)$. Then equation 11 gives the pure extension

$$\sqrt{2}\, u_1 = \gamma (X_1 + X_2) , \; u_2 = u_3 = 0 .$$

The specimen is a section of the circular cylindrical rod of arbitrary radius whose axis is along the x_1 axis. The section is cut off by the planes $X_1 + X_2 = 0$, $X_1 + X_2 = l$ (say). From equation 15

$$2\mathbf{t}_{(\mathbf{B})} . \mathbf{A} = \gamma (C_{1111} + C_{1212}) + 2\gamma C_{1112} .$$

Now C_{1111} has been determined under (i) and C_{1212} under (ii). Thus, by measuring $\mathbf{t}_{(\mathbf{B})} . \mathbf{A}$ and γ, C_{1112} can be determined.

Appropriate **A** and **B** to determine the remainder of the constants in (iv) are given by

$$C_{1113} : \mathbf{A} = (1, 0, 0), \; \sqrt{2}\, \mathbf{B} = (1, 0, 1);$$
$$C_{2223} : \mathbf{A} = (0, 1, 0), \; \sqrt{2}\, \mathbf{B} = (0, 1, 1);$$
$$C_{2221} : \mathbf{A} = (0, 1, 0), \; \sqrt{2}\, \mathbf{B} = (1, 1, 0);$$
$$C_{3331} : \mathbf{A} = (0, 0, 1), \; \sqrt{2}\, \mathbf{B} = (1, 0, 1);$$
$$C_{3332} : \mathbf{A} = (0, 0, 1), \; \sqrt{2}\, \mathbf{B} = (0, 1, 1).$$

(v) To obtain C_{1213} let $\mathbf{A} = (1, 0, 0)$, $\sqrt{2}\, \mathbf{B} = (0, 1, 1)$, $\mathbf{A} . \mathbf{B} = 0$. Then by equation 11 the simple shear is given by

$$\sqrt{2}\, u_1 = \gamma (X_1 + X_2) , \; u_2 = 0 , \; u_3 = 0 .$$

The specimen is the cube whose faces are $X_1 = 0, 1$; $X_2 + X_3 = 0 , \sqrt{2}$; $X_2 - X_3 = 0 , \sqrt{2}$.

From equation 15 it follows that

$$2\mathbf{t}_{(\mathbf{B})} \cdot \mathbf{A} = \gamma\,(C_{1212} + C_{1313}) + 2\gamma\, C_{1213}\,.$$

Now C_{1212} and C_{1313} have been obtained under (ii). Thus to obtain C_{1213}, only $\mathbf{t}_{(\mathbf{B})} \cdot \mathbf{A}$ and γ need be measured.

Suitable choices of $\mathbf{A}$ and $\mathbf{B}$ to obtain C_{2321} and C_{3132} are to take for

$$C_{2321} : \mathbf{A} = (0, 1, 0)\,, \quad \sqrt{2}\,\mathbf{B} = (1, 0, 1)\,,$$
$$C_{3132} : \mathbf{A} = (0, 0, 1)\,, \quad \sqrt{2}\,\mathbf{B} = (1, 1, 0)\,.$$

(vi) To obtain C_{1123} let $\sqrt{2}\,\mathbf{A} = (1, 1, 0)$, $\sqrt{2}\,\mathbf{B} = (1, 0, 1)$. Then from equation 11

$$2u_1 = X_1 + X_3\,, \quad 2u_2 = X_1 + X_3\,, \quad u_3 = 0\,.$$

The specimen is a section of a circular cylindrical rod, of arbitrary radius, whose axis is along $(1/\sqrt{2}, 1/\sqrt{2}, 0)$. The section is cut off by the parallel planes $X_1 + X_3 = 0, l$ (say). From equation 15

$$4\mathbf{t}_{(\mathbf{B})} \cdot \mathbf{A} = \gamma\, C_{1111} + \gamma\,(C_{1212} + C_{1313} + C_{2323}) + 2\gamma\,(C_{1112} + C_{1113}) + 2\gamma\,(C_{1213} + C_{1323} + C_{2312}) + 2\gamma\, C_{1123}.$$

The first term on the right is obtained in (i), the next group of terms under (ii), the next group under (iv) and the fourth group under (v).

A set of values of $\mathbf{A}$ and $\mathbf{B}$ suitable for the determination of C_{2231} and C_{3312} is given by

$$C_{2231} : \sqrt{2}\,\mathbf{A} = (0, 1, 1)\,, \quad \sqrt{2}\,\mathbf{B} = (1, 1, 0)\,,$$
$$C_{3312} : \sqrt{2}\,\mathbf{A} = (1, 0, 1)\,, \quad \sqrt{2}\,\mathbf{B} = (0, 1, 1)\,.$$

4. Concluding Remarks

(i) In the procedure outlined here, various particular choices of **A** and **B** have been made from the infinity of possibilities available. The motivation has been in each case, a desire for mathematical simplicity and economy of effort in determining the constants. Obviously twenty-one other choices of A and B would have done equally well provided they were suitable chosen.

(ii) Two basic types of specimen have been used. A cube is used in simple shear experiments and circular cylindrical rods for the other cases. The cubes are taken to be of unit length, and so the same apparatus may be used for the various simple shear experiments. For the other experiments the same apparatus may be used in each case provided the cylindrical specimens have the same diameter.

(iii) There are other ways of measuring the elastic constants. Suppose $C_{\alpha\beta\gamma\epsilon}$ is to be measured, where α, β, γ, ϵ are fixed numbers chosen from the range 1 2, 3. Let the deformation

$$u_i = \mathrm{K}\,\delta_{i\gamma}\, X_\epsilon\,, \qquad (16)$$

be produced in the body. Here K is a constant. Then

$$\frac{\partial u_k}{\partial X_l} = \mathrm{K}\,\delta_{k\gamma}\,\delta_{\epsilon l}\,, \quad t_{ij} = \mathrm{K}\, C_{ij\gamma\epsilon}\,, \qquad (17)$$

so that

$$t_{\alpha\beta} = \mathrm{K}\, C_{\alpha\beta\gamma\epsilon}\,. \qquad (18)$$

Thus for the deformation (equation 16) the measurement of $t_{\alpha\beta}$ and K provides $C_{\alpha\beta\gamma\epsilon}$.

This approach can be used for all α, β, γ, ϵ and provides a simple way of interpreting the elastic constants. However, different types of measurement have to be made in each case.

Also, it has been assumed throughout that $\mathbf{t}_{(\mathbf{B})} \cdot \mathbf{A}$ is to be measured. If $\mathbf{t}_{(B)} \cdot \mathbf{B}$ is measured instead, it provides a single constant in a number of cases. From equation 17

$$\mathbf{t}_{(\mathbf{B})} \cdot \mathbf{B} = t_{ij}\, B_i B_j = \gamma\, C_{ijkl}\, B_i\, B_j\, A_k\, B_l\,. \qquad (19)$$

Putting $B_i = \delta_{i\alpha}$, $A_k = \delta_{k\beta}$, this leads immediately to $C_{\alpha\alpha\beta\alpha}$. For example, suppose $\alpha = 2$, $\beta = 1$. Then equation 11 gives the simple shear

$$u_1 = \gamma\, X_2\,, \quad u_2 = u_3 = 0\,, \qquad (20)$$

and from equation 19

$$\mathbf{t}_{(\mathbf{B})} \cdot \mathbf{B} = t_{22} = \gamma\, C_{2212}\,.$$

Here t_{22} is a normal stress associated with the simple shear. There is no reason for supposing that this will be a small quantity compared with t_{12}, so there need not be any difficulty in making accurate measurements.

(iv) A method similar to that used here would work equally well in the case of more general theories, where the number of independent constants exceeds twenty-one.

References

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